THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 8

1. Let $f(x, y) = xy$.

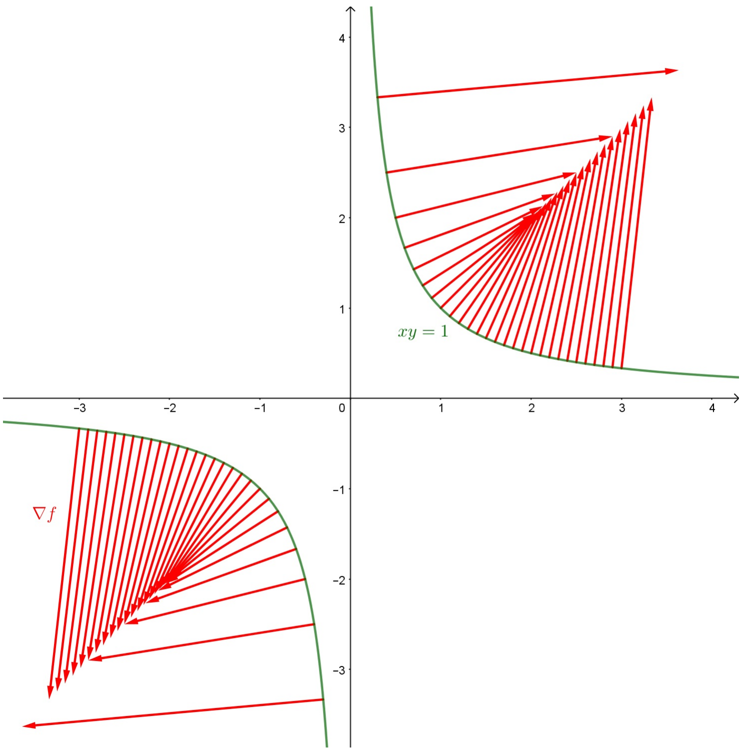
   (a) Draw the level set $L_1(f)$.

   (b) Find ∇f and draw ∇f restricted on $L_1(f)$.

   Ans:

   (a) $L_1(f) = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ which is a hyperbola.

   (b) $\nabla f(x, y) = (y, x)$.



(Remark: You can observe that ∇f is normal to the level set.)

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function such that all second partials of f are continuous. Suppose that $\mathbf{v} = (v_1, v_2, v_3)$ is a unit vector, express $\nabla_{\mathbf{v}}(\nabla_{\mathbf{v}} f)$ in terms of the components of $\mathbf{v}$ and the second partials of f.

   What is the interpretation of this quantity for a moving observer?

   Ans:

   Let $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Thus

$$
\begin{aligned}
\nabla f &= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \\
\nabla_{\mathbf{v}} f &= \mathbf{v} \cdot \nabla f \\
&= v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y} + v_3 \frac{\partial f}{\partial z} \\
\nabla(\nabla_{\mathbf{v}} f) &= \left(v_1 \frac{\partial^2 f}{\partial x^2} + v_2 \frac{\partial^2 f}{\partial x \partial y} + v_3 \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{i} + \left(v_1 \frac{\partial^2 f}{\partial y \partial x} + v_2 \frac{\partial^2 f}{\partial^2 y} + v_3 \frac{\partial^2 f}{\partial y \partial z} \right) \mathbf{j} + \left(v_1 \frac{\partial^2 f}{\partial z \partial x} + v_2 \frac{\partial^2 f}{\partial y \partial z} + v_3 \frac{\partial^2 f}{\partial^2 z} \right) \mathbf{k} \\
\nabla_{\mathbf{v}}(\nabla_{\mathbf{v}} f) &= \mathbf{v} \cdot \nabla(\nabla_{\mathbf{v}} f) \\
&= v_1^2 \frac{\partial^2 f}{\partial x^2} + 2v_1 v_2 \frac{\partial^2 f}{\partial x \partial y} + 2v_1 v_3 \frac{\partial^2 f}{\partial x \partial z} + v_2^2 \frac{\partial^2 f}{\partial y^2} + 2v_2 v_3 \frac{\partial^2 f}{\partial y \partial z} + v_3^2 \frac{\partial^2 f}{\partial^2 z}
\end{aligned}
$$

Furthermore, $\nabla_{\mathbf{v}}(\nabla_{\mathbf{v}} f)$ gives the second time derivative of the quantity f as measured by an observer moving with constant velocity $\mathbf{v}$.

3. Find the Taylor series generated by the following functions at given points and write down your answers in summation notation.

   (a) $f(x) = \cos x$ at $x = \pi/2$;

   (b) $f(x) = \ln(1+x)$ at $x = 0$;

   (c) $f(x) = e^x$ at $x = 1$.

   Ans:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}(x - \frac{\pi}{2})^{2n-1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$

4. By considering the Taylor series generated by e^x and $\cos x$ at $x = 0$, find the Taylor polynomials of degree 3 generated by the following functions at $x = 0$.

   (a) $e^x \cos x$;

   (b) $e^{\cos x}$;

   (c) $\dfrac{e^x}{\cos x}$.

   Ans:

   (a)

$$\begin{aligned} T(x) &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 - \frac{x^2}{2} + \cdots\right) \\ &= 1 + x + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{6} - \frac{1}{2}\right)x^3 + \cdots \\ \therefore \quad T_3(x) &= 1 + x - \frac{x^3}{3}. \end{aligned}$$

   (b)

$$\begin{aligned} T(x) &= 1 + \cos x + \frac{\cos^2 x}{2!} + \frac{\cos^3 x}{3!} + \cdots \\ &= 1 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{1}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 \\ &\quad + \frac{1}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^3 + \cdots \\ \therefore \quad T_3(x) &= \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) + \left[-\frac{x^2}{2} + \frac{1}{2!}\cdot 2 \cdot (-\frac{x^2}{2}) + \frac{1}{3!}\cdot 3 \cdot (-\frac{x^2}{2}) + \cdots\right] \\ &= e - \frac{x^2}{2}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) \\ &= e - \frac{e}{2}x^2. \end{aligned}$$

(c) Suppose $T(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, then

$$\begin{aligned}
e^x &= (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)\cos x \\
1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots &= \left(1 - \frac{x^2}{2} + \cdots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) \\
\therefore & \begin{cases} 1 \cdot c_0 = 1 \\ 1 \cdot c_1 = 1 \\ c_2 - \frac{c_0}{2} = \frac{1}{2} \\ c_3 - \frac{c_2}{2} = \frac{1}{6} \end{cases} \\
& c_0 = 1,\ c_1 = 1,\ c_2 = 1,\ c_3 = \frac{2}{3}.
\end{aligned}$$

Therefore, $T_3(x) = 1 + x + x^2 + \frac{2}{3}x^3$.

5. (a) Find the Taylor polynomial $P_2(x)$ of degree 2 generated by the function $\sqrt[3]{1+x}$.

   (b) Hence, approximate $\sqrt[3]{1.3}$ and show that the error of your approximation is less that 2×10^{-3}.

Ans:

(a) $P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{x}{3} - \frac{x^2}{9}$

(b) We can approximate $\sqrt[3]{1.3}$ by $P_2(0.3) = 1.09$. The error of this approximation is $E_2(0.3)$ which can be estimated by

$$\begin{aligned}
E_2(0.3) &= \frac{f'''(c)}{3}(0.3)^3 \qquad \text{for some } c \in (0, 0.3) \\
&= \frac{5}{81}(1+c)^{-8/3}\left(\frac{3}{10}\right)^3 \\
&< \frac{5}{81}\left(\frac{3}{10}\right)^3 \\
&= \frac{1}{600} \\
&< 2 \times 10^{-3}
\end{aligned}$$

6. Let $f(x) = \ln(1-x)$ for $x < 1$.

   (a) Find the Taylor series generated by $f(x)$ at $x = 0$.

   (b) Write down the Taylor polynomial $T_3(x)$ of degree 3 generated by $f(x)$ at $x = 0$ and the Lagrange remainder $E_3(x)$.

   (c) Hence, approximate $\ln 0.9$ and show that the error of your approximation is less than $\frac{1}{4 \times 9^4}$.

Ans:

(a) Taylor series generated by $f(x)$ at $x = 0$ is

$$\sum_{n=1}^{\infty} -\frac{1}{n}x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

(b) $T_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$ and

$E_3(x) = \frac{f^{(4)}(c)}{4!}x^4$ for some c lying between 0 and x (and so c depends on x).

Therefore,

$$\begin{aligned}
f(x) &= T_3(x) + E_3(x) \\
\ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{f^{(4)}(c)}{4!}x^4
\end{aligned}$$

(c) By putting $x = 0.1$, we have

$$\begin{aligned}
\ln(1-0.1) &= -0.1 - \frac{0.1^2}{2} - \frac{0.1^3}{3} + \frac{f^{(4)}(c)}{4!}(0.1^4) \\
\ln 0.9 &= -\frac{79}{750} + \frac{(\frac{-3!}{(1-c)^4})}{4!}(0.1)^4 \\
\left|\ln 0.9 - (-\frac{79}{750})\right| &= \frac{1}{4(1-c)^4}(0.1)^4 \\
&< (\frac{1}{4})(\frac{10}{9})^4(\frac{1}{10})^4 \\
&= \frac{1}{4\times 9^4}
\end{aligned}$$

Note that $0 < c < 0.1$, so $\frac{1}{1-c} < \frac{1}{0.9} = \frac{10}{9}$

$\ln 0.9$ can be approximated by $-\frac{79}{750} \approx -0.1053333$ with absolute error less than $\frac{1}{4\times 9^4}$.

7. Let $f(x)$ is a polynomial of degree $n > 0$ and let $a \in \mathbb{R}$.

(a) If $P_n(x)$ is the Taylor polynomial of degree n generated by $f(x)$ at $x = a$, show that $f(x) = P_n(x)$.

(b) Suppose that $f(a) = f'(a) = \cdots = f^{(r-1)}(a) = 0$ and $f^{(r)}(a) \neq 0$, where $1 \le r \le n$.
Prove that $(x-a)$ is a factor of $f(x)$ with multiplicity r, i.e. $f(x) = (x-a)^r g(x)$ for some polynomial $g(x)$ such that $g(x)$ is not divisible by $x - a$.

(c) By using the result in (b), factorize $x^5 - 7x^4 + 19x^3 - 25x^2 + 16x - 4$.

Ans:

(a) Let $x \in \mathbb{R}$. By Taylor's theorem, there exists c that lies on the open interval between x and a such that

$$f(x) - P_n(x) = E_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} = 0.$$

We have the last equality since f is only a polynomial of degree n.

(b) By (a), we have

$$\begin{aligned}
f(x) &= P_n(x) \\
&= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(r-1)}(a)}{(r-1)!}(x-a)^{r-1} + \\
&\qquad \frac{f^{(r)}(a)}{r!}(x-a)^r + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= \frac{f^{(r)}(a)}{r!}(x-a)^r + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{(By assumption)} \\
&= (x-a)^r \left(\frac{f^{(r)}(a)}{r!} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^{n-r}\right) \\
&= (x-a)^r g(x)
\end{aligned}$$

where $g(x) = \frac{f^{(r)}(a)}{r!} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^{n-r}$ which is a polynomial.

Note that $g(a) = \frac{f^{(r)}(a)}{r} \neq 0$, therefore $g(x)$ is not divisible by $x - a$.

(c) Note that $f(1) = f'(1) = f''(1) = 0$ but $f'''(1) \neq 0$, so $f(x)$ is divisible by $(x-1)^3$ (but not $(x-1)^k$ for any $k \ge 4$).
Similarly, note that $f(2) = f'(2) = 0$, so $f(x)$ is divisible by $(x-2)^2$ (but not $(x-2)^k$ for any $k \ge 3$).
Furthermore, $f(x)$ is of degree 5 and so $f(x) = A(x-1)^3(x-2)^2$.
The coefficient of x^5 of $f(x)$ is 1, so $A = 1$ and $f(x) = (x-1)^3(x-2)^2$.